ALGEBRAIC INDEPENDENCE OF SOME SERIES AND CONTINUED FRACTIONS IN THE FIELDS OF FORMAL POWER SERIES

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Outline

1. Introduction
   - The field of formal power series over $\mathbb{F}_q$
   - Algebraic independence
   - Continued fraction of power series
2. Algebraic independence of some series
3. Algebraic independence criteria

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Let $\mathbb{F}_q$ be a finite field of characteristic p. If T is an indeterminate, we denote by $\mathbb{F}_q(T)$ the field of rational function and $\mathbb{F}_q((T^{-1}))$ the field formal power series over $\mathbb{F}_q$.
For a nonzero power series α in $\mathbb{F}_q((T^{-1}))$ we can write

$$\alpha = \sum_{n \geqslant n_0} c_n T^{-n} \quad \text{with} \quad n_0 \in \mathbb{Z},\ c_n \in \mathbb{F}_q \quad \text{and} \quad c_{n_0} \neq 0.$$

An ultrametric absolute value is defined over this field by $|0| = 0, |\alpha| = q^{n_0}$.

The analogy

Let $\mathbb{F}_q$ be field and T a formal indeterminate. We have the following analogy :

$$\begin{array}{ccc} \mathbb{Z} & & \mathbb{F}_q[T] \\ \bigcap & & \bigcap \\ \mathbb{Q} & \approx & \mathbb{F}_q(T) \\ \bigcap & & \bigcap \\ \mathbb{R} & & \mathbb{F}_q((T^{-1})). \end{array}$$

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Definition

$\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{K}$ *are algebraically independent if there is non trivial polynomial with coefficient in* $\mathbb{K}$ *such that* $P(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$.

examples

1) π , $\exp(\pi)$, and $\Gamma(\frac{1}{4})$ are algebraically independent over $\mathbb{Q}$.

2) π and $\exp(1)$ are transcendental,it is not know whether $\{\pi, \exp(1)\}$ is algebraically independent over $\mathbb{Q}$.

3) $\alpha = \sqrt{2}$ and $\beta = \sqrt{8}$ they are individually algebraic, but they are algebraically dependent because $P(\sqrt{2}, \sqrt{8}) = 0$ where $P(x, y) = x^2 - \frac{1}{4}y^2$.

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Continued fraction of power series

As in classical continued fraction theory of real numbers, if $\alpha \in \mathbb{F}_q((T^{-1}))$, then we can write

$$\alpha = a_0 + 1/(a_1 + 1/(a_2 + 1/(\ldots = [a_0, a_1, a_2, \ldots],$$

where $a_i \in \mathbb{F}_q[T]$, with $\deg(a_i) \geqslant 1$ for any $i \geqslant 0$.
The sequence $(a_i)_{i \geqslant 0}$ is called the partial quotients of α

Lemma

Let $\xi = [a_0, a_1, \ldots]$ *be in* $\mathbb{F}_q((T^{-1}))$ *and* $(p_n/q_n)_{n\geq 0}$ *be the convergent sequence of* ξ. *Then the following hold:*

1. $\frac{p_n}{q_n} = [a_0, a_1, \ldots a_n] \quad (n \geq 0)$,
2. $\gcd(p_n, q_n) = 1 \quad (n \geq 0)$,
3. $|q_n| = |a_1||a_2| \cdots |a_n| \quad (n \geq 1)$,
4. $|\xi - \frac{p_n}{q_n}| = \frac{1}{|q_n||q_{n+1}|} = \frac{1}{|a_{n+1}||q_n|^2} \quad (n \geq 0)$
5. $\xi = \frac{\xi_{n+1}p_n + p_{n-1}}{\xi_{n+1}q_n + q_{n-1}}$, *where* $\xi_{n+1} = [a_{n+1}, a_{n+2}, \ldots]$.

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Algebraic independance of continued fractions in $\mathbb{F}_q((T^{-1}))$

Theorem (Algebraic independance)

Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}_q((T^{-1}))$. Assume that there are polynomials $p_{N,j}, q_{N,j}(\neq 0)$ $(N = 1, 2, \ldots; 1 \leq j \leq n)$ in $\mathbb{F}_q[T]$ with

$$q_{N,j}\alpha_j \neq p_{n,j}, \quad M_{N,j} = \max\{|p_{N,j}|, |q_{N,j}|\} \to +\infty \quad (N \to +\infty)$$

such that, if $n \geq 2$.

$$\lim_{N\to+\infty} \frac{|\alpha_{j-1} - \frac{p_{N,j-1}}{q_{N,j-1}}|}{|\alpha_j - \frac{p_{N,j}}{q_{N,j}}|} = 0 \text{ for } j = 2, 3, \ldots n.$$

Assume further that for each positive real number M,

Theorem (Algebraic independance)

an $L = L_0 \in \mathbb{N}$ *such that:*

$$|\alpha_j - \frac{p_{p,j}}{q_{N,j}}| \leq \frac{1}{(M_{N,1} M_{N,2} \cdots M_{N,j})^M} \quad (N \geq L_0; j = 1, 2, \dots, n).$$

Then $\alpha_1, \dots, \alpha_n$ *are algebraically independent over* $\mathbb{F}_q(T)$.

Objective

▽ **Our aim is to prove the algebraic independence of some series in $\mathbb{F}_q((T^{-1}))$ and the analogue of Bundschuh'criteria of algebraic independance in the fields of Laurent series.**

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Theorem (Ben Bechir)

Let $\chi_1, \chi_2, \ldots, \chi_m \in \mathbb{F}_q[T]$ *with* $|\chi_{i-1}| > |\chi_i|$ *for all* $2 \leq j \leq m$ *;*δ *a positif real number and* (b_n) *be a sequence of positive real numbers where* $b_{n+1} = b_n^{2+\delta}$ *for all* $n \geq 1$ *;* $b_0 = 1$ *and* $b_1 = \sqrt{2}$ *. Let*

$$S_j = \sum_{n=0}^{\infty} \chi_j^{-b_n}$$

for all $1 \leq j \leq m$*, Then,* $S_1, S_2, \ldots, S_m$ *are algebraically independent over* $\mathbb{F}_q(T)$.

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Lemma (Ben Bechir)

Let $\xi_j = [a_{0,j}, a_{1,j}, \ldots, a_{n,j}, \ldots]$ *for* $j = 2, \ldots, m$ *be in* $\mathbb{F}_q((T^{-1}))$ *;* $\xi_{n,j} = [a_{0,j}, a_{1,j}, \ldots, a_{n,j}]$ *and* $(p_{n,j}); (q_{n,j})$ *be sequence of convergents of* ξ_j*. Then:*

1. *if there exist a real number* $r > 1$*, such that* $\frac{|a_{n,j-1}|}{|a_{n,j}|} > r$ *for all* $j = 2, \ldots, m$ *and* $n \geq 1$*, then we get*

$$|q_{n,j-1}| > r^{\frac{n}{2}} |q_{n,j}| > |q_{n,j}|. \qquad (1)$$

2. *Let* (γ_n) *a real increasing sequence* > 1*. If* $|a_{n,j}| > |a_{n-1,j}|^{\gamma_n}$ *for all* $n \geq 2$ *then for any* $\epsilon > 0$ *we obtain:* $|q_{n,j}| < |a_{n,j}|^{\frac{\gamma_1}{\gamma_1 - 1} + \epsilon}$.

Theorem (Ben Bechir)

Let $\xi_1, \xi_2, \ldots, \xi_m \in \mathbb{F}_q((T^{-1}))$ *which are defined by the continued fraction* $\xi_j = [a_{0,j}, a_{1,j}, \ldots, a_{n,j}, \ldots]$ *for all* $1 \leq j \leq m$. *We suppose that there exist a real number* $r > 1$, *and* (γ_n) *a real increasing sequence* > 1. *Such that* $|\frac{a_{n,j-1}}{a_{n,j}}| > r$ *and* $|a_{n+1,m}| > |a_{n,1}^{\gamma_n}|$ *for all* $n \geq 3$.
Then $\xi_1, \xi_2, \ldots, \xi_m$ *are algebraically independent over* $\mathbb{F}_q(T)$.

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