On bi-periodic Fibonacci numbers and continued fraction expansion

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   - Continued fraction
   - The bi-periodic Fibonacci sequence
2. Objective
3. New results
   - Bi-periodic Fibonacci sequence and periodic continued fraction
   - Rational approximation
   - On the series of reciprocals
4. Perspective

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The algorithm of continued fraction

Given a real number x, we have $x = [x] + \{x\}$. If x is not an integer, then $\{x\} \neq 0$ and setting $x_1 := 1/\{x\}$, we have :

$$x = [x] + \frac{1}{x_1}.$$

Again, if x_1 is not an integer, then $\{x_1\} \neq 0$ and setting $x_2 := 1/\{x_1\}$ we get :

$$x = [x] + \cfrac{1}{[x_1] + \cfrac{1}{x_2}}.$$

This process stops if for some i it occurs $\{x_i\} = 0$, otherwise it continues forever.Writing $a_0 = [x]$ and $a_i = [x_i]$ for $i \geq 1$, this sequence $(a_i)_{i \geqslant 0}$ is called the partial quotients.
We obtain the socalled continued fraction expansion of x.

The algorithm of continued fraction

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

Which from now on we will write it with the more succinct notation :

$$x = [a_0, a_1, a_2, a_3, \cdots]$$

Define p_n and q_n by

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2} & \text{, for } n \geq 1 \\ q_n = a_n q_{n-1} + q_{n-2} & \text{, for } n \geq 1 \end{cases}$$

with : $p_{-1} = 1,\ p_0 = a_0,\ q_{-1} = 0$ and $q_0 = 1$.
Then $[a_0, a_1, \ldots, a_n] = p_n/q_n$, which is called the n-th convergent.

Definition

A continued fraction is periodic if there exist integers $k \geq 0$ such that : $a_{n+k} = a_n$, for all $n \geq 0$. We write

$$x = [a_0, a_1, \cdots, a_r, \overline{a_{r+1}, \cdots, a_{r+k}}].$$

▶ Quadratic irrational $\Longleftrightarrow$ Periodic continued fraction expansion.

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- Quadratic irrational $\Longleftrightarrow$ Periodic continued fraction expansion.

Example

The continued fraction of the golden ratio $\phi = \dfrac{1+\sqrt{5}}{2}$ is :

$$\frac{1+\sqrt{5}}{2} = 1 + \frac{\sqrt{5}-1}{2} = 1 + \cfrac{1}{\cfrac{2}{\sqrt{5}-1}}$$

$$= 1 + \cfrac{1}{\cfrac{\sqrt{5}+1}{2}} = 1 + \cfrac{1}{1 + \cfrac{\sqrt{5}-1}{2}}$$

Then, $\phi = \dfrac{1+\sqrt{5}}{2} = [1, 1, \cdots, 1, \cdots] = [\overline{1}]$.

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Definition

Let a and b be two fixed real numbers. The bi-periodic Fibonacci sequence $\{F_n^{(a,b)}\}_{n=0}^{\infty}$ is defined by :

$$F_n^{(a,b)} = \begin{cases} aF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)} & \text{for } n \text{ even} \\ bF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)} & \text{for } n \text{ odd} \end{cases}$$

with, $F_0^{(a,b)} = 0$ and $F_1^{(a,b)} = 1$.

Note that the sequences $\{F_n^{(a,b)}\}$ can be presented by :

$$F_n^{(a,b)} = a^{\xi(n+1)} b^{\xi(n)} F_{n-1}^{(a,b)} + F_{n-2}^{(a,b)},$$

with

$$\xi(m) = \begin{cases} 0 & \text{for } m \quad even \\ 1 & \text{for } m \quad odd \end{cases}$$

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Special case

- If $a = b = 1$, the definition reduces to the classical Fibonacci sequence :

$$F_n = F_{n-1} + F_{n-2},$$

with, $F_0 = 0$ and $F_1 = 1$.

- The Pell's sequence defined by

$$P_n = 2P_{n-1} + P_{n-2}$$

is $\{F_n^{(a,b)}\}$ with $a = b = 2$.

- If $a = b = k$, the definition reduces to the k-Fibonacci sequence :

$$F_n = kF_{n-1} + F_{n-2}.$$

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Edson Marcia, and Omer Yayenie (2009).

▶ M. Edson and O. Yayenie introduced and studied the bi-periodic Fibonacci sequence $\{F_n^{(a,b)}\}_{n=0}^{\infty}$.

▶ We establish several properties of this sequence using periodic continued fractions.

▶ In particular, we describe some properties of the Lagrange spectrum.

▶ We also determine the value of certain infinite series involving reciprocals of the product of two bi-periodic Fibonacci sequences.

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Theorem (Abbes, Ayadi, and Saadaoui)

Let $(p_n/q_n)_{n\geq 0}$ be the sequence of convergents of $\lambda = [0, \underbrace{b, a, \ldots}_{n}] = [0, \overline{b, a}]$. Then for all $n \geq 0$

$$p_n = F_n^{(a,b)},$$

and

$$\begin{cases} q_n = F_{n+1}^{(a,b)} & \text{for } n \text{ even} \\ q_n = ba^{-1}F_{n+1}^{(a,b)} & \text{for } n \text{ odd} \end{cases}$$

- The denominators of the n-th convergent to λ admit the compact form $q_n = b^{\xi(n)}a^{-\xi(n)}F_{n+1}^{(a,b)}, \quad n \geq 1.$

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Corollary 1

We obtain the following continued fraction representations :

$$\begin{cases} \dfrac{F_n^{(a,b)}}{F_{n+1}^{(a,b)}} = [0, \underbrace{b, a, \ldots}_{n}] = [0, \overline{b, a}^{\frac{n}{2}}] & \text{if } n \text{ is even} \\ \dfrac{F_n^{(a,b)}}{ba^{-1}F_{n+1}^{(a,b)}} = [0, \underbrace{b, a, \ldots}_{n}] = [0, \overline{b, a}^{\frac{n-1}{2}}, b] & \text{if } n \text{ is odd} \end{cases}$$

Special case

- If $a = b = 1$, and use the fact that $\phi = \lim_{n \to +\infty} \dfrac{F_{n+1}}{F_n}$. Then

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Corollary 2

For $n \geq 1$, we have the Cassini's identity for the bi-periodic Fibonacci sequence :

$$a^{1-\xi(n)}b^{\xi(n)}F_{n-1}^{(a,b)}F_{n+1}^{(a,b)} - a^{\xi(n)}b^{1-\xi(n)}(F_n^{(a,b)})^2 = a(-1)^n.$$

- We use the fact that

$$q_{2n}p_{2n-1} - q_{2n-1}p_{2n} = 1 \text{ and } q_{2n+1}p_{2n} - q_{2n}p_{2n+1} = -1.$$

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Dirichlet's theorem

Johann Peter Gustav Lejeune Dirichlet (1842).
For every irrational real number x, there exist infinitely many rational approximations satisfying :

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Hurwitz's theorem

Adolf Hurwitz (1891).

Hurwitz's theorem improves Dirichlet's classical result, which asserts the existence of infinitely many rational approximations satisfying

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}\, q^2}.$$

Waldschmidt's theorem

Michel Waldschmidt(2008).

Waldschmidt showed that equality is asymptotically attained for the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$, since

$$\lim_{n\to+\infty} F_{n-1}^2 \left|\phi - \frac{F_n}{F_{n-1}}\right| = \frac{1}{\sqrt{5}},$$

where (F_n) is the classical Fibonacci sequence, that is $F_n = F_n^{(1,1)}$, and $\phi = \lim_{n\to+\infty} \frac{F_n}{F_{n-1}}$.

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Proposition

- Given an irrational real number x, define its Lagrange constant by

$$l(x) = \limsup_{p,q\to\infty} \frac{1}{q^2\left|x - \frac{p}{q}\right|}.$$

- The Lagrange spectrum is then defined as

$$L = \{\, l(x) < \infty \mid x \in \mathbb{R} \setminus \mathbb{Q} \,\}.$$

- In this language, Hurwitz's theorem states that $\min L = \sqrt{5}$.

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Theorem (Abbes, Ayadi, and Saadaoui)

Let λ be the quadratic irrational number defined by $\lambda(a,b) = [0,b,a,b,a,\ldots] = [0,\overline{b,a}]$.. Then

$$\lim_{n\to+\infty} b^{\xi(n)}a^{-\xi(n)}\left(F_{n+1}^{(a,b)}\right)^2 \left|\lambda - \frac{F_n^{(a,b)}}{b^{\xi(n)}a^{-\xi(n)}F_{n+1}^{(a,b)}}\right| = \frac{a}{\sqrt{a^2b^2+4ab}}$$

Special case

- If $a = b = 1$, we obtain the Waldschmidt's limit.

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Corollary

For any positive integers a and b, the Lagrange spectrum L contains the value

$$\frac{1}{a}\sqrt{a^2b^2 + 4ab}.$$

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Proposition

Let $x = [a_0, a_1, \ldots]$ be a real number with convergents $(p_n/q_n)_{n\geq 0}$. For all $n \geq 0$, we have :

1. $q_{n+1}p_n - q_np_{n+1} = (-1)^{n+1}$.
2. $q_{n+1}p_{n-1} - q_{n-1}p_{n+1} = (-1)^n a_{n+1}$.
3. $q_np_{n+3} - q_{n+3}p_n = (-1)^n(a_{n+2}a_{n+3} + 1)$.

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Lemma

- Let $x = [a_0, a_1, \cdots, a_n, \cdots]$ and $(p_n/q_n)_{n\geq 0}$ be its sequence of convergents. Then, for all $n \geq 0$

1. $$\frac{p_n}{q_n} = a_0 + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{q_{k-1}q_k}.$$
2. $$\frac{p_{2n}}{q_{2n}} = a_0 + \sum_{k=1}^{n} \frac{a_{2k}}{q_{2k-2}q_{2k}}.$$
3. $$\frac{p_{3n}}{q_{3n}} = a_0 + \sum_{k=1}^{n} \frac{(-1)^{k-1}(a_{3k}a_{3k-1}+1)}{q_{3k-3}q_{3k}}.$$

Theorem (Abbes, Ayadi, and Saadaoui)

Let λ be the quadratic irrational number defined by $\lambda(a,b) = [0,b,a,b,a,\ldots] = [0,\overline{b,a}]$..We have that

1. $$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{F_n^{(a,b)} F_{n+1}^{(a,b)}} = ba^{-1}\lambda.$$
2. $$\sum_{n=1}^{+\infty} \frac{1}{F_{2n}^{(a,b)} F_{2n+2}^{(a,b)}} = a^{-2} - a^{-2}b\lambda.$$
3. $$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{F_{3n}^{(a,b)} F_{3n+3}^{(a,b)}} = \frac{ba^{-1}}{(ba+1)}(\lambda - a(ba+1)^{-1}).$$

Special case

- If $a = b = 1$, then $\lambda(a,b) = \lambda(1,1) = [0,1,1,1,1,\ldots] = [0,\overline{1}]$. We have that

1. $$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{F_n F_{n+1}} = \lambda = \frac{1}{\phi}.$$
2. $$\sum_{n=1}^{+\infty} \frac{1}{F_{2n} F_{2n+2}} = 1 - \lambda = 1 - \frac{1}{\phi}.$$

Theorem (Abbes, Ayadi, and Saadaoui)

We have that

$$\sum_{n=1}^{+\infty} \frac{a^{\xi(n+1)} b^{\xi(n)}}{F_n^{(a,b)} F_{n+2}^{(a,b)}} = \frac{1}{a}.$$

Special case

- If $a = b = 1$, we obtain the series of the reciprocals of the product of two Fibonacci numbers :

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- If $a = b = 1$, we obtain the series of the reciprocals of the product of two Fibonacci numbers :

$$\sum_{n=1}^{+\infty} \frac{1}{F_n F_{n+2}} = 1.$$

Corollary

For $k \geq 1$, we have :

1. $$\sum_{n=1}^{+\infty} \frac{1}{F_n^{(k,k)} F_{n+2}^{(k,k)}} = \frac{1}{k^2},$$
2. $$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{F_n^{(k,k)} F_{n+1}^{(k,k)}} = -\phi_{k,2},$$
3. $$\sum_{n=1}^{+\infty} \frac{1}{F_{2n}^{(k,k)} F_{2n+2}^{(k,k)}} = k^{-2} + k^{-1}\phi_{k,2},$$
4. $$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{F_{3n}^{(k,k)} F_{3n+3}^{(k,k)}} = -\frac{1}{k^2+1}\left(\frac{k}{k^2+1} + \phi_{k,2}\right).$$

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? Perspective

- One possible extension is to investigate analogous questions for other bi-periodic sequences such that the bi-periodic Lucas sequence.

$$L_n^{(a,b)} = \begin{cases} bL_{n-1}^{(a,b)} + L_{n-2}^{(a,b)} & \text{for } n \text{ even} \\ aL_{n-1}^{(a,b)} + L_{n-2}^{(a,b)} & \text{for } n \text{ odd} \end{cases}$$

with, $L_0^{(a,b)} = 1$ and $L_1^{(a,b)} = a$.

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Thank you for your attention