The Digital Expansion of Squares and Primes

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Christian Mauduit 1959–2019

Summary

- ★ Thue-Morse sequence
- ★ Gelfond-Problems
- ★ Sarnak Conjecture
- ★ Combined Problems
- ★ Zeckendorf Expansion
- ★ Methods
- ★ Open Problems

★ Thue-Morse sequence

Thue-Morse sequence $(t_n)_{n\geqslant 0}$:

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$$0$$

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01

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Thue-Morse sequence $(t_n)_{n\geqslant 0}$:

0110

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Thue-Morse sequence $(t_n)_{n\geqslant 0}$:

01101001

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0110100110010110

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$$t_0 = 0, \qquad t_{2^n+k} = 1-t_k \quad (0 \leqslant k < 2^n)$$

$$\boxed{t_n = s_2(n) \bmod 2}$$

$$n = \sum_{i=0}^{\ell-1} \varepsilon_i(n) q^i \qquad \varepsilon_i(n) \in \{0,1,\ldots,q-1\}, \qquad s_q(n) = \sum_{i=0}^{\ell-1} \varepsilon_i(n)$$

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$$t_0 = 0, \qquad t_{2^n+k} = 1 - t_k \quad (0 \leqslant k < 2^n) \qquad \text{or} \qquad t_{2k} = t_k,\ t_{2k+1} = 1 - t_k$$

$$\boxed{t_n = s_2(n) \bmod 2}$$

$$n = \sum_{i=0}^{\ell-1} \varepsilon_i(n) q^i \qquad \varepsilon_i(n) \in \{0, 1, \ldots, q-1\}, \qquad s_q(n) = \sum_{i=0}^{\ell-1} \varepsilon_i(n)$$

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The letters 0 and 1 appear with asymptotic frequency $\frac{1}{2}$.

★ Thue-Morse sequence

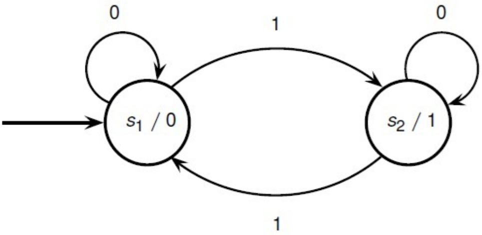
- TM sequence is **not periodic** and **cubeless**.
- TM sequence is **almost periodic**:
  Every appearing consecutive block appears infinitely many times with bounded gaps.
- **Subword complexity is linear**: $p_k \leqslant \frac{10}{3}k$
  p_k ... subword complexity (*number of different consecutive blocks of length k that appear in the TM sequence*).
- **Zero topological entropy** of the corresponding dynamical system:

$$h = \lim_{k\to\infty} \frac{1}{k} \log p_k = 0$$

- **Linear subsequences** $(t_{an+b})_{n\geqslant 0}$ have the same properties.
- The TM sequence and their linear subsequences are **automatic sequences**.

★ Thue-Morse sequence

Automaton that generates the Thue-Morse sequence:
$t_n = \sum_{j\geqslant 0} \varepsilon_j(n) \bmod 2 = s_2(n) \bmod 2$



★ Background 1: Gelfond Problems

1. $q_1, q_2, \ldots, q_d \geqslant 2$, $(q_i, q_j) = 1$ for $i \neq j$, $(m_j, q_j - 1) = 1$:

$$\boxed{\#\{n < N : s_{q_j}(n) \equiv \ell_j \bmod m_j,\ 1 \leqslant j \leqslant d\} = \frac{N}{m_1 \cdots m_d} + O(N^{1-\eta})}$$

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D., Mauduit, Rivat 2011: large bases, Spiegelhofer: $P(x) = x^3$

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★ Background 1: Gelfond Problems

Thue-Morse sequence $(t_n)_{n\geqslant 0}$ along primes:

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10 0 1 1 1 0 1 0 0 1 1 1 0 · · ·

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$$10\ \ 0\ \ 1\qquad 1\ \ 1\qquad 0\ \ 1\qquad 0\qquad\quad 0\ \ 1\qquad\quad 1\qquad 1\ \ 0\ \cdots$$

Theorem (Mauduit, Rivat 2010)

Suppose that $q \geqslant 2$ and $m \geqslant 2$ withe $(m, q-1) = 1$. Then for all ℓ

$$\#\{primes\ p < N : s_q(p) \equiv \ell \bmod m\} = \frac{\pi(N)}{m} + O(N^{1-\eta})$$

for some $\eta > 0$.

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Theorem (Mauduit, Rivat 2009)

Suppose that $q \geqslant 2$ *and* $m \geqslant 2$ *withe* $(m, q-1) = 1$*. Then for all* ℓ

$$\#\{n < N : s_q(n^2) \equiv \ell \bmod m\} = \frac{N}{m} + O(N^{1-\eta})$$

for some $\eta > 0$*.*

★ Background 1: Gelfond Problems

Theorem (D., Mauduit and Rivat, 2011)

For every $d \geqslant 2$ *there exists* $q_0(d) \geqslant 2$ *such that for all prime* $\boxed{q \geqslant q_0(d)}$ *and all integer polynomials* $P(x)$ *of degree* d *(where the leading coefficient if coprime to* q*)*

$$\#\{1 \leqslant n < N : s_q(P(n)) \equiv a \bmod m\} = \frac{N}{m} + O\left(N^{1-\eta}\right)$$

for some $\eta > 0$ *and all integers* m *with* $(m, q-1) = 1$*.*

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Theorem (Spiegelhofer)

$$\#\{1 \leqslant n < N : s_2(n^3) \equiv 0 \bmod 2\} = \frac{N}{2} + O\left(N^{1-\eta}\right)$$

for some $\eta > 0$*.*

★ Background 1: Gelfond Problems

Definition

A sequence $(u_n)_{n\geqslant 0}$ is called a *q*-**automatic sequence**, if u_n is the output of an automaton when the input is the *q*-ary expansion of *n*.

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A sequence $(u_n)_{n\geqslant 0}$ is called a q**-automatic sequence**, if u_n is the output of an automaton when the input is the q-ary expansion of n.

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Lemma

For every automatic sequence $(u_n)_{n\geqslant 0}$ *the* **densities**

$$\mathrm{dens}(u_n; a) = \lim_{N\to\infty} \frac{1}{N} \#\{n \leqslant N : u_n = a\}$$

or, if not, the **logarithmic densities**

$$\mathrm{logdens}(u_n; a) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{n\leqslant N} \frac{1}{n} \mathbf{1}_{u_n = a}$$

exist.

★ Background 1: Gelfond Problems

Theorem (Adamczewski+D.+Müllner 2020)

For every automatic sequence $(u_n)_{n\geqslant 0}$ *the logarithmic densities*

$$\boxed{\mathrm{logdens}(u_{p_n}, a)}$$

of the subsequence along **prime numbers** *and the logarithmic densities*

$$\boxed{\mathrm{logdens}(u_{n^2}, a)}$$

of the subsequence along **squares** *exist and are computable. Furthermore, the densities* $\boxed{\mathrm{dens}(u_{p_n}, a)}$ *and* $\boxed{\mathrm{dens}(u_{n^2}, a)}$ *exist if and only if the densities* $\mathrm{dens}(u_n, a)$ *exist.*

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This is the most general result related to the Gelfond problems so far.

★ Background 2: Sarnak Conjecture

A bounded complex valued sequence $u(n)$, $n \in \mathbb{N}$, is said to be **deterministic** if for every $\epsilon > 0$ the set $\{(u(n+1), \ldots, u(n+m)) : n \in \mathbb{N}\}$ can be covered by $O(\exp(o(m)))$ balls of radius ϵ (as $m \to \infty$).

For example, if $\boxed{u(n) := f(T^n x)}$ for a minimal topological dynamical system (X, T) with **zero topological entropy** (and a continuous function f) then $u(n)$ is deterministic.

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In particular **automatic sequences** $u(n)$ are deterministic.
And every sequence $u(n) := f(T^n x)$, where $x \in X$ and (X, T) is generated from an **automatic sequence** is deterministic.

★ Background 2: Sarnak Conjecture

Definition (Möbius-function)

$$\mu(n) = \begin{cases} 1 & n = 1, \\ (-1)^r & n = p_1 p_2 \cdots p_r, \\ 0 & n \text{ is not square free.} \end{cases}$$

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$\mu(3) = \mu(5) = -1$, $\mu(6) = \mu(35) = 1$

★ Background 2: Sarnak Conjecture

Theorem

$$\sum_{n=1}^{N} \mu(an+b) = o(N) \qquad (N \to \infty)$$

About **half** *of the (square-free) numbers of the form $an+b$ with $n \leqslant N$ have an* **even number** *of has prime divisors and the other* **half** *has an* **odd number** *of prime divisors.*

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The relation $\sum_{n=1}^{N} \mu(n) = o(N)$ is equivalent to the **prime number theorem** $\pi(x) \sim x/\log x$ and the general property $\sum_{n=1}^{N} \mu(an+b) = o(N)$ is equivalent to the **Dirichlet prime number theorem** $\pi(x;a,m) \sim x/(\varphi(m)\log x)$.

★ Background 2: Sarnak Conjecture

A sequence $u(n)$ is **orthogonal to the Möbius function** $\mu(n)$ if

$$\sum_{n\le N} \mu(n)u(n) = o(N) \qquad (N \to \infty).$$

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Conjecture (Sarnak conjecture)

Every deterministic bounded complex valued sequence $u(n)$, $n \in \mathbb{N}$ is orthogonal to the Möbius function.

★ Background 2: Sarnak Conjecture

The following sequences are orthogonal to $\mu(n)$:

- For constant sequences.
- For periodic sequences.
- For quasiperiodic sequences such as $f(n) = F(\alpha n \bmod 1)$ for some continuous F.
- For nilsequences (Green+Tao).
- For horocycle flows (Bourgain+Sarnak+Ziegler).
- For the Thue-Morse sequence (Dartyge-Tenenbaum).
- For the Rudin-Shapiro sequence (Tao, Mauduit+Rivat).
- For all **automatic sequences** (Müllner).
- ...

★ Background 2: Sarnak Conjecture

Definition (Von Mangoldt Λ-Function)

$$\Lambda(n) = \begin{cases} \log p & n = p^k \text{ for some prime } p \text{ and some } k \geqslant 1, \\ 0 & \text{else.} \end{cases}$$

★ Background 2: Sarnak Conjecture

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$$\Lambda(n) = \begin{cases} \log p & n = p^k \text{ for some prime } p \text{ and some } k \geqslant 1, \\ 0 & \text{else.} \end{cases}$$

$$\sum_{n \leqslant N} \Lambda(n) u_n \approx \log N \sum_{p \leqslant N,\, p \in \mathbb{P}} u_p$$

★ Background 2: Sarnak Conjecture

The analysis of

$$\sum_{n\leqslant N} \Lambda(n) G(u(n))$$

which leads to the sum $\sum_{p\leqslant N} G(u(p))$ and to distribution properties of the subsequence $u(p)$, $p \in \mathbb{P}$, is completely similar to the analysis of

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which is needed for the Sarnak conjecture.

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which is needed for the Sarnak conjecture.

Thus, there is a natural relation between the **Gelfond problem on primes** and the **Sarnak conjecture**.

★ Background 2: Sarnak Conjecture

Theorem (Müllner 2017)

For every automatic sequence $(u_n)_{n\geqslant 0}$ *one has*

$$\sum_{n\leq N} \mu(n) u_n = o(N) \qquad (N \to \infty).$$

For every primitive automatic sequence $(u_n)_{n\geqslant 0}$ *one has*

$$\sum_{n\leqslant N} \Lambda(n) u_n = cN + o(N)$$

for some real c.

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for some real c.

An automatic sequence u_n is called primitive if the associated dynamical system (X,T) is minimal, that is, $\overline{\{T^n(x) : n\geqslant 0\}} = X$ for all $x\in X$, or if the associated minimal automaton is strongly connected.

★ Background 2: Sarnak Conjecture

Theorem (Adamczewski+D.+Müllner 2020)

Every automatic sequence $(u_n)_{n\geqslant 0}$ *satisfies*

$$\sum_{n\leqslant N}\frac{1}{n}\Lambda(n)u_n = c\log N + o(\log N)$$

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Theorem (D.+Lemańczyk+Müllner+Rivat 2025)

Suppose that $(u_n^{(j)})_{n\geqslant 0}$ *are primitive* q_j*-automatic sequences for* $1\leqslant j\leqslant d$ *such that* $q_1,\ldots,q_d$ *are pairwise coprime. Then*

$$\sum_{n\leq N}\mu(n)u_n^{(1)}u_n^{(2)}\cdots u_n^{(d)} = o(N) \qquad (N\to\infty).$$

★ Variants and Combined Problems

Theorem (D., Mauduit and Rivat, 2009)

Suppose that $(q, k-1) = 1$*. Then*

$$\#\{\textit{primes } p < N : s_q(p) = k\}$$

$$= \frac{q-1}{\varphi(q-1)} \frac{\pi(N)}{\sqrt{2\pi\sigma_q^2 \log_q N}} \left(\exp\left(-\frac{(k-\mu_q \log_q N)^2}{2\sigma_q^2 \log_q N} \right) + O((\log N)^{-\frac{1}{2}+\varepsilon}$$

where

$$\mu_q := \frac{q-1}{2}, \quad \sigma_q^2 := \frac{q^2-1}{12}.$$

★ Variants and Combined Problems

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$$\#\{primes\ p < N : s_q(p) = k\}$$

$$= \frac{q-1}{\varphi(q-1)} \frac{\pi(N)}{\sqrt{2\pi\sigma_q^2 \log_q N}} \left(\exp\left(-\frac{(k-\mu_q \log_q N)^2}{2\sigma_q^2 \log_q N} \right) + O((\log N)^{-\frac{1}{2}+\varepsilon}$$

where

$$\mu_q := \frac{q-1}{2}, \quad \sigma_q^2 := \frac{q^2-1}{12}.$$

$$\#\{\text{primes } p < 2^{2k} : s_2(p) = k\} \sim \frac{2^{2k}}{\sqrt{2\pi}\log 2\, k^{\frac{3}{2}}}$$

★ Variants and Combined Problems

Theorem (D.+Mauduit+Rivat 2019)

The Thue-Morse sequence $t_n = s_2(n) \bmod 2$ *along squares is a* **normal sequence***:*

$$\#\{n < N : t_{n^2} = \ell_0,\ t_{(n+1)^2} = \ell_1, \ldots, t_{(n+d-1)^2} = \ell_{d-1}\} \sim \frac{N}{2^d}$$

for every $d \geqslant 1$ *and for every* $\ell_0, \ldots, \ell_{d-1} \in \{0, 1\}$*.*

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Theorem (D.+Mauduit+Rivat+Spiegelhofer 2022)

Let t_n *denote the Thue–Morse sequence. Then we have, as* $N \to \infty$*,*

$$\sum_{n<N} \mu(n) t_{n^2} = o(N).$$

★ Variants and Combined Problems

Theorem (D.+Mauduit+Rivat 2020)

Suppose that q_1 and q_2 are different prime numbers such that $(q_1-1, m_1) = (q_1-1, m_2) = 1$. Then

$$\#\{\textit{primes } p < N : s_{q_1}(p) \equiv \ell_1 \bmod m_1,\ s_{q_2}(p) \equiv \ell_2 \bmod m_2\} \sim \frac{\pi(N)}{m_1 m_2}$$

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There is now a generalization to several q's by Müllner, Spiegelhofer, and Shubin (ongoing work).

★ Variants and Combined Problems

Theorem (D.+Rivat)

Suppose that $(q-1,m)=1$ *and that* q *is a sufficiently large prime number. Then*

$$\#\{primes\ p < N : s_q(p^2) \equiv a \bmod m\} = \frac{\pi(x)}{m} + O\left(x^{1-\frac{c'}{\log q}\|(q-1)\gamma\|^2}\right)$$

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This result **combines** the Gelfond problesm for primes and squares. The method does now work, for example, for $q = 2$.

★ Zeckendorf Expansion

Zeckendorf expansion

($F_0 = 0, F_1 = 1, F_{k+1} = F_k + F_{k-1}$ Fibonacci numbers)

$$\boxed{n = \sum_{j\geqslant 2} \varepsilon_{Z,j}(n) F_j}, \quad \varepsilon_{Z,j}(n) \in \{0,1\},\ \varepsilon_{Z,j}(n)\varepsilon_{Z,j+1}(n) = 0.$$

Zeckendorf Sum-of-digits function

$$s_Z(n) = \sum_{j\geqslant 2} \varepsilon_{Z,j}(n)$$

★ Zeckendorf Expansion

Theorem (D.+Müllner+Spiegelhofer 2018)

We have for non-integers α

$$\sum_{n<N} \mu(n) e(\alpha s_Z(n)) = o(N).$$

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Theorem (D.+Müllner+Spiegelhofer 2025)

We have for non-integers α *(and some* $c > 0$*)*

$$\sum_{n\leqslant N} \Lambda(n) \mathrm{e}\big(\alpha\, s_Z(n)\big) \ll (\log N)^5 N^{1-c\|\alpha\|^2}.$$

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$$\#\{\text{primes } p \leqslant N : s_Z(p) \equiv a \bmod m\} = \frac{\pi(N)}{m} + O(N^{1-\eta})$$

★ Zeckendorf Expansion

Theorem (D.+Müllner+Spiegelhofer 2025)

For each $\varepsilon > 0$*, we have*

$$\#\{p \leqslant x : s_Z(p) = k\} = \frac{\pi(x)}{\sqrt{2\pi\sigma^2 \log_\gamma x}} \left(e^{-\frac{(k-\mu \log_\gamma x)^2}{2\sigma^2 \log_\gamma x}} + O\left((\log x)^{-\frac{1}{2}+\varepsilon}\right) \right)$$

uniformly for all integers $k \geqslant 0$*, where*

$$\mu = \frac{1}{\gamma^2+1} \quad \text{and} \quad \sigma^2 = \frac{\gamma^3}{(\gamma^2+1)^3},$$

$\pi(x)$ *denotes the number of primes* $\leqslant x$*, and* $\log_\gamma x = \log x / \log \gamma$.

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In particular, for every sufficiently large integer k there exists a prime number p with

$$s_Z(p) = k.$$

★ Proof Methods

- Discrete Fourier techniques
- Exponential sum estimates (van der Corput inequality etc.)
- Digit detection techniques
- Diophantine approximation (Baker's theorem, *p*-adic subspace theorem)
- Discrepancy estimates
- Daboussi-Kátai-criterion
- Vaughan's method
- Level of distribution and Gowers norm estimates

★ Discrete Fourier techniques

$$F_\lambda(h) = \frac{1}{q^\lambda} \sum_{0 \leqslant u < q^\lambda} e\left(\alpha s_q(u) - hq^{-\lambda}\right)$$

Lemma

We have uniformly for all h:

$$|F_\lambda(h)| \leqslant C_1 q^{-c_1 \|(q-1)\alpha\|^2 \lambda}$$

and

$$\sum_{0 \leqslant h < q^\lambda} |F_\lambda(h)| \leqslant C_2 q^{\eta_q \lambda}$$

for some positive constants C_1, c_1, C_2 *and some* $0 < \eta_q < \frac{1}{2}$.

★ Daboussi-Kátai-criterion

Lemma

Let $f(n)$ be a bounded sequence such that

$$\sum_{n\leqslant N} f(pn)\overline{f(qn)} = o(N)$$

for all distinct prime numbers p, q. Then

$$\sum_{n\leqslant N} \mu(n)f(n) = o(N).$$

There is a similar criterion by Bourgain-Sarnak-Ziegler.

★ Vaughan's method

Lemma

Let $f:\mathbb{N}\to\mathbb{C}$ *such that* $|f(n)|\leqslant 1$ *for all* $n\geq 1$. *For all* $N,U,V\geqslant 2$ *such that* $UV\leq N$ *we have*

$$\sum_{n\leq N}\Lambda(n)\,f(n)\ll U+(\log N)\sum_{t\leq UV}\max_{w}\left|\sum_{w\leq r\leq N/t}f(rt)\right|$$

$$+\sqrt{N}(\log N)^3\max_{\substack{U\leq M\leq N/V\\ V\leq q_2\leq N/M}}\left(\sum_{V<q_1\leq N/M}\left|\sum_{\substack{M<m\leq 2M\\ m\leq\min(N/q_1,N/q_2)}}f(mq_1)\overline{f(mq_2)}\right|\right)^{1/2},$$

with an absolute implied constant.

★ Level of distribution and Gowers norm

Level of distribution of the Thue-Morse sequence: For all $\varepsilon > 0$ we have

$$\sum_{1 \le d \le x^{1-\varepsilon}} \max_{\substack{y,z \ge 0 \\ z-y \le x}} \max_{0 \le a < d} \left| \sum_{\substack{y \le n < z \\ n \equiv a \bmod d}} (-1)^{s_2(n)} \right| = \mathcal{O}(x^{1-\eta})$$

for some $\eta > 0$ depending on ε; the level of distribution equals **1.**

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Gowers norm of the Thue-Morse sequence:

$$\sum_{0\le n,r_1,\ldots,r_m<2^\rho} \mathrm{e}\left(\alpha \sum_{\varepsilon\in\{0,1\}^m} s_\rho(n+\varepsilon\cdot r)\right) \ll 2^{(m+1)\rho\cdot(1-c\|\alpha\|^2)}$$

for all $\rho \ge 0$, where $\varepsilon \cdot r = \sum_{1\le i\le m} \varepsilon_i r_i$ and s_ρ is the *truncated sum-of-digits function* in base 2.

★ Open Problems

- Gelfond-Problem for general integer polynomials and ALL $q \geqslant 2$.
- Sarnak conjecture for general morphic sequences (fixed points of general finite substitutions)
- Precise behavior of $S(N) = \sum_{n \leqslant N} e(\alpha s_2(n))$
- Distribution of $s_Z(n^2) \bmod m$
- More general numeration systems
- ...

Thank you very much for your attention!