Superzeta Functions and τ-Li Coefficients for a class of L-Functions

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Plan

1. **Introduction**
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Selberg class: definition

Alte Selberg

i) Dirichlet series

For $\mathrm{Re}(s) > 1,\ F(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}$

ii) Analytic continuation

There exists an integer $m \in \mathbb{N}$ such that $(s-1)^m F(s)$ is an entire function of finite order. The smallest such number is denoted by m_F and called the polar order of F.

iii) Functional equation

$\phi_F(s) = \omega\overline{\phi_F(1-\overline{s})},$ where $\phi_F(s) = F(s) Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j),$
with $Q_F > 0, r \geq 0, \lambda_j > 0, |w| = 1$ and $\mathrm{Re}\mu_j \geq 0, j = 1, \ldots, r.$

v) Euler product

$$\log F(s) = \sum_{n=1}^{+\infty} \frac{b(n)}{n^s},$$

where $b(n) = 0$ for all $n \neq p^m$ with $m \geq 1$ (p prime) and $b_F(n) \ll n^\theta$, for some $\theta < \frac{1}{2}$.

iv) Ramanujan hypothesis

$\forall\ \epsilon > 0, \quad a(n) = O(n^\epsilon).$

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The extended Selberg class $S^\#$ is a class of functions satisfying axioms (i), (ii) and (iii).

Notations

- For any function F in the Selberg class $\mathcal{S}$, we define the degree of F as:

$$d = d_F = 2\sum_{j=1}^{r} \lambda_j.$$

- We denote by $\mathcal{S}_d$ the set of functions of $\mathcal{S}$ with fixed degree d.
- The logarithmic derivative of $F(s)$ has also the following Dirichlet series expansion

$$-\frac{F'}{F}(s) = \sum_{n=1}^{+\infty} \Lambda_F(n) n^{-s}, \qquad \Re e(s) > 1,$$

where $\Lambda_F(n) = b(n)\log n$ is an analogue of the Von-Mangoldt function $\Lambda(n)$.

Example

The Riemann zeta function

The Riemann zeta function is defined for $\sigma > 1$ by :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} (1 - p^{-s})^{-1}.$$

It has the functional equation

$$\phi(s) = \phi(1-s)$$

where

$$\phi(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s),$$

where Γ is the gamma function.

Structure of the Selberg Class

- $S_0 = \{1\}$ et $S_d = \emptyset$ for $0 < d < 1$ (Conrey & Ghosh 1993).
- $S_1 = \{\zeta,\ L(s + iA, \chi)\}$ (Kaczorowski & Perelli 1999) and (Soundararajan 2005)
- $S_d = \emptyset$ for $1 < d < \frac{5}{3}$ (Kaczorowski & Perelli 2002).
- $S_d = \emptyset$ for $1 < d < 2$ (Kaczorowski & Perelli 2010).

Generalized Riemann Hypotheis

Thorem (Conrey et Ghosh 1993)

If $F \in \mathcal{S}$ Then F does not vanish for $\sigma > 1$.

The zeros of a function in $\mathcal{S}$ are :

-The trivial zeros

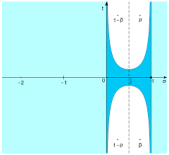
$$-\frac{n+\mu_j}{\lambda_j}, \textit{where } n \in IN \textit{ and } 1 \leq j \leq r.$$

- The non-trivial zeros in the critical strip $0 \leq \sigma \leq 1$

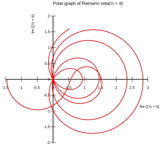
Generalized Riemann Hypotheis

For every function in the Selberg class the analogue of the Riemann hypothesis holds, that is all nontrivial zeros lie on the critical line $Re(s) = \frac{1}{2}$.

Distribution of zeros



Distribution of zeros of the Riemann zeta function



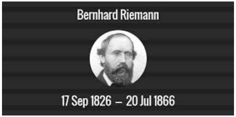
The Riemann zeta function

Riemann-Von Mangoldt formula

Let $N_F(T) = |\{F(\rho) = 0; 0 \leq \beta \leq 1 \text{ and } 0 \leq \gamma \leq T\}|$. Then

$$N_F(T) = 2T\left(\frac{d_F}{4\pi}(\log T - 1) + \frac{1}{4\pi}\log(\lambda Q_F^2)\right) + c_2 \log T + O\left(\frac{1}{T}\right).$$

Riemann Hypothesis



Lagarias (2002)

$$(HR) \quad \Leftrightarrow \quad \sum_{d|n} 1 \leq H_n + e^{H_n} \log H_n, \; H_n = \sum_{j=1}^{n} \frac{1}{j}.$$

Balazard, Saias et Yor (1999)

$$(HR) \quad \Leftrightarrow \quad \int_{-\infty}^{\infty} \frac{\log|\zeta(1/2+it)|}{1+4t^2} dt = 0.$$

Volchkov (1995)

$$(HR) \quad \Leftrightarrow \quad \int_{0}^{\infty} \frac{t \arg \zeta(1/2+it)}{(1/4+t^2)^2} dt = \pi \frac{\gamma-3}{2},$$ où γ est la constante d'Euler.

The class $\mathcal{S}^{\sharp\flat}$

Throughout this talk, we focus on the class $\mathcal{S}^{\sharp\flat}$ of functions satisfying axioms (i), (ii) and the following two axioms:

(iii') (Functional equation) The function F satisfies the functional equation $\xi_F(s) = w\overline{\xi_F(1-\bar{s})}$, where

$$\xi_F(s) = s^{m_F}(1-s)^{m_F}F(s)Q_F^s\prod_{j=1}^{r}\Gamma(\lambda_j s+\mu_j) = \gamma_F(s)F(s)$$

with $Q_F > 0$, $r \geq 0$, $\lambda_j > 0$, $|w| = 1$, $\mathrm{Re}\mu_j > -\frac{1}{4}$, $\mathrm{Re}(\lambda_j + 2\mu_j) > 0$, $j = 1, \ldots, r$.

(v') (Euler sum) The logarithmic derivative of the function F possesses a Dirichlet series representation

$$\frac{F'}{F}(s) = -\sum_{n=2}^{\infty}\frac{c_F(n)}{n^s}, \quad \mathrm{Re}(s) > 1.$$

Li criterion

Let $\xi(s) = s(s-1)\Gamma(s/2)\pi^{-s/2}\zeta(s)$. The Li coefficients are defined by:

$$\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n}\left[s^{n-1}\log\xi(s)\right]_{s=1} = \sum_{\rho;\xi(\rho)=0}\left[1-(1-\frac{1}{\rho})^n\right].$$

We have

$$\frac{\phi'(s)}{\phi(s)} = \sum_{n=0}^{+\infty}\lambda_{n+1}s^n, \text{où } \phi(s) := \xi\left(\frac{1}{1-s}\right) = \xi\left(\frac{s}{s-1}\right).$$

Theorem (Li 1997)

$$(HR) \Leftrightarrow \forall\ n \in \mathbb{N},\ \lambda_n \geq 0.$$

- X- J. Li (2004) : Dirichlet L functions.
- S. Omar et K. Mazhouda (2007) : Selberg class.
- K. Mazhouda et Smajlović (2019) : Function fields.

τ-Li coefficients

Motivated by the work of Freitas (2005) in the case of the classical Riemann zeta function, Droll[1] in 2012 defined for real number $\tau \geq 1$ and positive integer n, the τ-Li coefficients $\lambda_F(n,\tau)$ for the function $F \in S^{\sharp\flat}$, with $0 \notin Z(F)$ in the case $\tau = 1$, are defined by

$$\lambda_F(n,\tau) = \sum_{\rho \in Z(F)} \left(1 - \left(\frac{\rho}{\rho - \tau}\right)^n\right), \tag{1}$$

where $Z(F)$ is the set of all non-trivial zeros of F and the sum is taken in the sense of the limit: $\lim_{T\to\infty} \sum_{|\mathrm{Im}(\rho)|\leq T}$. We shall call this type of convergence the * convergence.

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[1] A. D. Droll, Variations of Li's criterion for an extension of the Selberg class, PhD thesis, Queen's University Ontario, Canada, 2012. (available at http://qspace.library.queensu.ca/jspui/bitstream/1974/ 7352/1/Droll Andrew D 201207 PhD.pdf.)

Proposition (Droll, Thesis 2012)

Let $F \in S^{\sharp\flat}$ and assume that $0 \notin Z(F)$ in the case $\tau = 1$. Let $\tau \geq 1$ be a real number.

(a) The sum $\lambda_F(n,\tau)$ defined in (1) is * convergent for every positive integer n.

(b) We have

$$\Re(\lambda_F(n,\tau)) = \sum_{\rho \in Z(F)} \Re\left[1 - \left(\frac{\rho}{\rho - \tau}\right)^n\right],$$

where the sum on the right-hand side is absolutely convergent for every positive integer n.

(c)

$$\sum_{\rho \in Z(F)} \frac{1 + |\Re(\frac{\rho}{\tau})|}{(1 + |\frac{\rho}{\tau}|)^2} < \infty.$$

Proposition (Droll, Thesis 2012)

Let $\tau \geq 1$ be a real number and let $F \in S^{\sharp\flat}$, with $0 \notin Z(F)$ in the case $\tau = 1$. Denote by $d_F(n, z_0)$ the power series coefficients in the expansion of the logarithmic derivative of $\xi_F\left(\frac{1}{1-s}\right)$ around $z_0 \neq 1$ which is not a zero of $\xi_F\left(\frac{1}{1-s}\right)$, so that in some neighborhood of z_0 we have

$$\frac{d}{ds} \log \xi_F\left(\frac{1}{1-s}\right) = \sum_{n=0}^{\infty} d_F(n, z_0)(s - z_0)^n. \tag{2}$$

Then, for every positive integer n,

$$\lambda_F(n, \tau) = \frac{\tau}{(n-1)!} \left[\frac{d^n}{ds^n}(s^{n-1} \log \xi_F(s))\right]_{s=\tau} = \frac{1}{\tau^n} d_F\left(n-1, 1-\frac{1}{\tau}\right). \tag{3}$$

Droll (2012) proved that the non-negativity of $\Re(\lambda_F(n,\tau))$ for all positive integers n is equivalent to the assertion: all nontrivial zeros of $F \in S^{\sharp\flat}$ are in the strip $1-\tau/2 \leq \Re(s) \leq \tau/2$ (this is known as the τ-Li criterion and it is a generalization of the Li criterion stated above, but the τ-Li criterion produces only zero-free regions).

Arthmetic formula (Bucur et al. (2016)). For $\tau \geq 1$,

$$\begin{aligned} \lambda_F(n,\tau) &= m_F + n\tau \log Q_F + \sum_{k=1}^{n} \binom{n}{k} \tau^k \eta_F(k-1,\tau) \\ &\quad + \sum_{k=1}^{n} \binom{n}{k} \frac{\tau^k}{(k-1)!} \sum_{j=1}^{r} \lambda_j^k \psi^{(k-1)}(\lambda_j \tau + \mu_j) \end{aligned}$$

(4)

For $\tau > 1$,

$$\lambda_F(n,\tau) = m_F\left[2 + (-1)^{n+1}\left(\frac{1}{\tau-1}\right)^n\right] + n\tau\log Q_F$$
$$+\sum_{k=1}^{n}\binom{n}{k}\tau^k\gamma_F(k-1,\tau) + \sum_{k=1}^{n}\binom{n}{k}\frac{\gamma^k}{(k-1)!}\sum_{j=1}^{r}\lambda_j^k\psi^{(k-1)}(\lambda_j\tau+\mu_j),$$

where

$$\eta_F(k,\tau) = \frac{1}{k!}\frac{d^k}{dz^k}\left[\frac{F'}{F}(z) + \frac{m_F}{z-1}\right]_{z=\tau}, \quad \text{for} \quad \tau \geq 1, \ \text{and} \ \ \gamma_F(k,\tau) = \frac{1}{k!}\frac{d^k}{dz^k}\left[\frac{F'}{F}(z)\right]_{z=\tau}, \quad \text{for} \quad \tau > 1.$$

An asymptotic formula for the archimedean term $n\tau\log Q_F + \sum_{k=1}^{n}\binom{n}{k}\frac{\gamma^k}{(k-1)!}\sum_{j=1}^{r}\lambda_j^k\psi^{(k-1)}(\lambda_j\tau+\mu_j)$ was established by Droll (2012) in terms of $n\log n$ and n.

Similar asymptotic formula for the τ-Li coefficients associated to automorphic L-functions was given by Mazhouda in 2017.

In 1917 Mellin considered a class of zeta functions that are defined by using the zeros of the Riemann zeta function as the building block of new Dirichlet series. He established meromorphic continuation to the whole complex plane for a class of such functions including the function $\mathbf{Z}(s) = \sum_\rho \rho^{-s}$, where the sum is taken over all nontrivial zeros of the Riemann zeta function. Voros in 2010 has dedicated a monograph [2] on generalizations of such functions for which he gave the name superzeta functions. He introduced three types of such functions, called of the first, second or third kind, and discussed possible expansions to other classes of zeta functions.

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[2] A. Voros, Zeta functions over zeros of zeta functions, Lecture Notes of UMI, vol. 8, Springer, 2010.

Definition and notations. Let $F \in S^{\sharp\flat}$ and $z \in X = \{z \in \mathbb{C} : \forall \rho \in Z(F), (z-\rho) \notin \mathbb{R}_-\}$ (here and henceforth $\mathbb{R}_- = \{x \in \mathbb{R}, x \leq 0\}$). The superzeta function of the first kind associated to F is defined by

$$\mathcal{Z}_F(s,z) = \sum_{\rho \in Z(F)} (z-\rho)^{-s}$$

It is well defined by the absolutely convergent series for $\Re(s) > 1$. The function $\mathcal{Z}_F(s,z)$ can be meromorphically continued to the whole complex plane having only one simple pole at $s = 1$, of residue $-d_F/2$. The series $\sum_\rho \frac{1}{(z-\rho)}$ is convergent in the sense of $\lim_{T\to\infty} \sum_{|\mathrm{Im}(\rho)| \leq T}$ for all $z \in X$. Put

$$\mathcal{Z}_F^*(1,z) := \frac{\xi_F'}{\xi_F}(z) = \sum_\rho \frac{1}{(z-\rho)}. \tag{5}$$

One has

$$\frac{\xi_F'}{\xi_F}(1) = \mathcal{Z}_F^*(1,1) = m_F + \log Q_F + \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j + \mu_j) + \gamma_F(0).$$

Here, $\gamma_F(k)$ denote the coefficients in the Taylor (Laurent) series expression of $\frac{F'}{F}(s)$ at (possible pole) $s = 1$.

Proposition (Bllaca et al. 2026)

Let n be a positive integer and $F \in S^{\sharp\flat}$ such that $0 \notin Z(F)$. Then, for $\tau \geq 1$ we have

$$\lambda_F(n,\tau) = n\tau\mathcal{Z}_F^*(1,\tau) + \sum_{m=2}^{n}(-1)^{m+1}\binom{n}{m}\tau^m\mathcal{Z}_F(m,\tau), \qquad (6)$$

where $\mathcal{Z}_F^*(1,\tau)$ is defined in (5).

From the definition of the τ-Li coefficients, for $m \geq 2$ we obtain

$$\mathcal{Z}_F(m,\tau) = \frac{1}{\tau^m}\sum_{n=1}^{m}(-1)^{n+1}\binom{m}{n}\lambda_F(n,\tau). \qquad (7)$$

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[3] Kajtaz H. Bllaca. Jawher Khmiri. Kamel Mazhouda. Bouchaïb Sodaïgui, *Superzeta functions and τ-Li coefficientsin the Selberg class*, Funct. Approx. Comment. Math. Advance Publication 1 - 21, 2026. https://doi.org/10.7169/facm/240906-20-10

Proof. We have

$$\begin{aligned}\lambda_F(n,\tau) &= \frac{\tau}{(n-1)!}\left[\frac{d^n}{ds^n}(s^{n-1}\log\xi_F(s))\right]_{s=\tau}\\ &= \sum_{m=1}^{n}\binom{n}{m}\frac{\tau^m}{(m-1)!}\left[\frac{d^m}{ds^m}\log\xi_F(s)\right]_{s=\tau}\\ &= n\tau\left[\frac{\xi_F'}{\xi_F}(z)\right]_{s=\tau}+\sum_{m=2}^{n}\binom{n}{m}\frac{\tau^m}{(m-1)!}\left[\frac{d^m}{ds^m}\log\xi_F(s)\right]_{s=\tau}.\end{aligned}$$

Let us write the following Hadamard product

$$\xi_F(s)=\xi_F(0)e^{b_F s}\prod_{\rho\in Z(F)}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}},$$

where $b_F=\frac{\xi_F'(0)}{\xi_F(0)}=-\sum^{*}_{\rho\in Z(F)}\frac{1}{\rho}$. By the absolute and uniform convergence of the infinite product on arbitrary closed discs, the factors satisfy the hypotheses of Lang's Lemma on any open disc. It follows that we may logarithmically differentiate the factors term by term to get

$$\frac{d}{ds}\log\xi_F(s)=b_F+\sum_{\rho\in Z(F)}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right),$$

where the sum over ρ is absolutely and uniformly convergent on any closed bounded region not containing ρ. Since each of the summands is analytic, then we can differentiate term by term repeatedly to find all of the derivatives of $\log\xi_F$ around any point which is not in $Z(F)$. Hence, for all $z\in\mathbb{C}$ different from zeros of ξ_F and $m\geq 2$ one has

$$\frac{d^m}{ds^m}\log\xi_F(z)=-(m-1)!(-1)^m\sum_{\rho\in Z(F)}\frac{1}{(z-\rho)^m}=(-1)^{m+1}(m-1)!\mathcal{Z}_F(m,z). \tag{8}$$

Using (5) and (8), we complete the proof.

Arithmetic formula

Theorem (Bllaca et al. 2026)

Let $F \in S^{\sharp\flat}$ such that $0 \notin Z(F)$. Then, for $\tau \geq 1$ and $m \geq 2$ we have

$$\mathcal{Z}_F(m,\tau) = (-1)^{m+1}\eta_F(m-1,\tau) + \frac{m_F}{\tau^m} + \frac{(-1)^{m+1}}{(m-1)!}\sum_{j=1}^{r}\lambda_j^m\psi^{(m-1)}(\lambda_j\tau+\mu_j),$$

where $\eta_F(m,\tau)$ denote the coefficients of the Laurent expansion of $\frac{F'}{F}(z) + \frac{m_F}{z-1}$ at $z=\tau$ and $\psi = \frac{\Gamma'}{\Gamma}$ denotes the digamma function. Moreover, for $\tau > 1$ and $m \geq 2$ we have

$$\begin{aligned}\mathcal{Z}_F(m,\tau) &= \frac{-1}{(m-1)!}\sum_{k=2}^{\infty}\frac{c_F(k)\log^{m-1}(k)}{k^\tau} \\ &\quad + m_F\left(\frac{1}{\tau^m} + \frac{1}{(\tau-1)^m}\right) + \frac{(-1)^{m+1}}{(m-1)!}\sum_{j=1}^{r}\lambda_j^m\psi^{(m-1)}(\lambda_j\tau+\mu_j).\end{aligned}$$

Proof. We start with the following expression of $\mathcal{Z}_F(m,z)$ given in terms of the derivatives: for $m \geq 2$, we have

$$\mathcal{Z}_F(m,\tau) = \frac{(-1)^{m+1}}{(m-1)!}\left[\frac{d^m}{ds^m}\log \xi_F(z)\right]_{z=\tau}.$$

Recall that $\xi_F(z) = z^{m_F}(z-1)^{m_F}F(z)Q_F^z\prod_{j=1}^r\Gamma(\lambda_j z+\mu_j)$. Therefore, for $m \geq 2$ we have

$$\begin{aligned}
\mathcal{Z}_F(m,\tau) &= \frac{(-1)^{m+1}}{(m-1)!}\left[\frac{d^m}{ds^m}\left(m_F(\log z+\log(z-1)) + z\log Q_F + \log F(z) + \sum_{j=1}^r \log\Gamma(\lambda_j z+\mu_j)\right)\right]_{z=\tau}\\
&= \frac{(-1)^{m+1}}{(m-1)!}\left[\frac{d^{m-1}}{ds^{m-1}}\left(m_F\left(\frac{1}{z}+\frac{1}{z-1}\right) + \log Q_F + \frac{F'}{F}(z) + \sum_{j=1}^r \lambda_j\frac{\Gamma'}{\Gamma}(\lambda_j z+\mu_j)\right)\right]_{z=\tau}\\
&= \frac{(-1)^{m+1}}{(m-1)!}\left[\frac{d^{m-1}}{ds^{m-1}}\left(\frac{F'}{F}(z)+\frac{m_F}{z-1}\right) + (-1)^{m+1}(m-1)!\frac{m_F}{z^m} + \sum_{j=1}^r \lambda_j^m\psi^{(m-1)}(\lambda_j z+\mu_j)\right]_{z=\tau},
\end{aligned}$$

where ψ denotes the digamma function. Since F has a pole of order m_F at $z=1$, $\frac{F'}{F}(z)+\frac{m_F}{z-1}$ is analytic for $z \geq 1$. Furthermore, for $F \in S^{\sharp\flat}$, we have $\frac{F'}{F}(z)$ is analytic for $z > 1$. Let us denote the following Laurent expansions at $z=\tau$ by:

$$\frac{F'}{F}(z)+\frac{m_F}{z-1} = \sum_{m=0}^{\infty}\eta_F(m,\tau)(z-\tau)^m, \quad \text{for} \quad \tau \geq 1,$$

and

$$\frac{F'}{F}(z) = \sum_{m=0}^{\infty}\gamma_F(m,\tau)(z-\tau)^m, \quad \text{for} \quad \tau > 1.$$

The coefficients $\gamma_F(m,\tau)$ are the τ-Euler-Stieltjes constants. Note that $\eta_F(m,\tau) = \gamma_F(m,\tau) + \frac{(-1)^m m_F}{(\tau-1)^{m+1}}$, for $\tau > 1$.
Therefore, we obtain for $\tau \geq 1$

$$\mathcal{Z}_F(m,\tau) = (-1)^{m+1}\eta_F(m-1,\tau) + \frac{m_F}{\tau^m} + \frac{(-1)^{m+1}}{(m-1)!}\sum_{j=1}^{r}\lambda_j^m \psi^{(m-1)}(\lambda_j\tau + \mu_j)$$

and for $\tau > 1$

$$\mathcal{Z}_F(m,\tau) = (-1)^{m+1}\gamma_F(m-1,\tau) + m_F\left(\frac{1}{\tau^m} + \frac{1}{(\tau-1)^m}\right) + \frac{(-1)^{m+1}}{(m-1)!}\sum_{j=1}^{r}\lambda_j^m \psi^{(m-1)}(\lambda_j\tau + \mu_j).$$

Since $\frac{F'}{F}(s) = -\sum_{n=2}^{\infty}\frac{c_F(n)}{n^s}$, then for $\tau > 1$ we have

$$\gamma_F(m,\tau) = \frac{(-1)^{m+1}}{m!}\sum_{k=2}^{\infty}\frac{c_F(k)\log^m(k)}{k^\tau}.$$

This completes the proof of the Theorem .
Another proof. for $m \geq 2$ we have $\mathcal{Z}_F(m,\tau) = \frac{1}{\tau^m}\sum_{n=1}^{m}(-1)^{n+1}\binom{m}{n}\lambda_F(n,\tau)$. Then

$$\begin{aligned}\mathcal{Z}_F(m,\tau) \quad &= \quad \frac{1}{\tau^m}\sum_{n=1}^{m}(-1)^{n+1}\binom{m}{n}(m_F + n\tau\log Q_F)\\ &+ \frac{1}{\tau^m}\sum_{n=1}^{m}(-1)^{n+1}\binom{m}{n}\sum_{k=1}^{n}\binom{n}{k}\tau^k\eta_F(k-1,\tau)\\ &+ \frac{1}{\tau^m}\sum_{n=1}^{m}(-1)^{n+1}\binom{m}{n}\sum_{k=1}^{n}\binom{n}{k}\frac{\tau^k}{(k-1)!}\sum_{j=1}^{r}\lambda_j^k\psi^{(k-1)}(\lambda_j\tau + \mu_j).\end{aligned}$$

Since $m \geq 2$ and

$$\sum_{n=1}^{m}(-1)^{n+1}\binom{m}{n}n = \delta_{m,1},$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$, we get

$$\begin{aligned} \mathcal{Z}_F(m,\tau) &= \frac{m_F}{\tau^m} + \sum_{n=1}^{m}\sum_{k=1}^{n}(-1)^{n+1}\binom{m}{n}\binom{n}{k}\tau^{k-m}\eta_F(k-1,\tau) \\ &\quad + \sum_{n=1}^{m}\sum_{k=1}^{n}(-1)^{n+1}\binom{m}{n}\binom{n}{k}\frac{\tau^{k-m}}{(k-1)!}\sum_{j=1}^{r}\lambda_j^k\psi^{(k-1)}(\lambda_j\tau+\mu_j), \end{aligned} \tag{9}$$

Let us use the following binomial identity

$$\sum_{n=1}^{m}\sum_{k=1}^{n}(-1)^{n+1}\binom{m}{n}\binom{n}{k}\tau^{k-m}B_k = \sum_{k=1}^{n}\sum_{n=k}^{m}(-1)^{n+1}\binom{m}{k}\binom{m-k}{n-k}\tau^{k-m}B_k = (-1)^{m+1}B_m. \tag{10}$$

This identity holds true since

$$\sum_{n=k}^{m}(-1)^{n+1}\binom{m}{k}\binom{m-k}{n-k} = 0$$

unless $k = m$. Now, (9) with the use of (10) prove the first assertion of Theorem. The same argument using the expression of the τ-Li coefficients for $\tau > 1$ and (10) prove the second assertion of Theorem.

Definition and notation

In order to define the superzeta function of the second kind we begin by pointing out that the absence of central symmetry $\rho \leftrightarrow 1-\rho$ in the set of the zeros of $F \in S^{\sharp\flat}$ implies that the zeros of F do not necessarily come in pairs $\rho_k = 1/2 \pm i\tau_k$ with $\Re(\tau_k) > 0$. That is the main reason why it is not possible to define for all $F \in S^{\sharp\flat}$ the superzeta function of the second kind associated to F introduced by Voros. However, if the coefficients $a_F(n)$ of the Dirichlet series representation of F are real numbers, then for all $n \in \mathbb{N}$, by the reflection principle, the zeros of F are symmetric with respect to the real line. (Actually, if ρ is a zero, then ρ, $1-\rho$ and $1-\overline{\rho}$ are zeros of F). Therefore, the zeros of F come in pairs $\rho_k = 1/2 \pm i\tau_k$ with $\Re(\tau_k) > 0$.

Now, let us denote by $S_{\mathbb{R}}^{\sharp\flat}$ the set of $F \in S^{\sharp\flat}$ with $a_F(n)$ are real numbers. Let $F \in S_{\mathbb{R}}^{\sharp\flat}$. We define the superzeta function of the second kind by

$$\mathfrak{Z}_F(s,t) = \sum_{k=1}^{\infty} \frac{1}{((\tau_k)^2 + t^2)^s}, \quad Re(s) > \frac{1}{2},$$

where $t \in \mathbb{C}$ such that $t^2 + (\tau_k)^2 \notin \mathbb{R}_-$ for all k.

One has the following Hadamard product

$$\xi_F(z) = \prod_{\rho \in Z(F)}{}^{*} \left(1 - \frac{z}{\rho}\right) := \lim_{T \longrightarrow \infty} \prod_{\substack{\rho \in Z(F) \\ |Im(\rho)| \leq T}} \left(1 - \frac{z}{\rho}\right). \qquad (11)$$

Moreover, if $\xi_F(1/2) \neq 0$ then

$$\xi_F(z) = \prod_{\substack{\rho \in Z(F) \\ Im(\rho) > 0}}{}^{*} \left(1 - \frac{z(1-z)}{\rho(1-\rho)}\right) := \lim_{T \longrightarrow \infty} \prod_{\substack{\rho \in Z(F) \\ 0 < Im(\rho) \leq T}} \left(1 - \frac{z(1-z)}{\rho(1-\rho)}\right). \qquad (12)$$

Remark. If $\xi_F(1/2) = 0$, instead of ξ_F we may take the function $E_F(z) = \xi_F(z)/(z - 1/2)^l$, where l is the multiplicity of the eventual zero of ξ_F at $z = 1/2$. Functions E_F and ξ_F have the same zeros. For this reason, we assume $\xi_F(1/2) \neq 0$.

Using $b_F(1) = 0$, one has (Odzak and Smajlovic, 2011)

$$(\log F(z))^{(n)} = O\left(\frac{1}{|z|^{N+1}}\right)$$

for all $N, n \in \mathbb{N}$ as $|z| \to \infty$ with $|\arg z| < \pi/2$.

Theorem (Bllaca et al. 2026)

Let $F \in S_{\mathbb{R}}^{\sharp\flat}$ such that $\xi_F(1/2) \neq 0$, and let $t \in \mathbb{C}$ satisfying $t^2 + (\tau_k)^2 \notin \mathbb{R}_-$ for all $k \in \mathbb{Z}$. The superzeta function $\mathfrak{Z}_K(s,t)$ has a meromorphic continuation to all $s \in \mathbb{C}$, with double poles at $1/2 - m$ $(m \in \mathbb{N} \cup \{0\})$.

We provide relations between superzeta functions of the second kind and the τ-Li coefficients when $\tau \geq 1$.

Theorem (Bllaca et al. 2026)

Let n be a positive integer, $\tau > 1$ and $F \in S^{\sharp\flat}$. Then

$$\begin{aligned}\lambda_F(n,\tau) \quad = \quad & n\tau\mathcal{Z}_F^*(1,\tau) \\ & + \sum_{m=2}^{n} \sum_{k \geq m/2}^{m} (-1)^{2m-k+1} \left(2 - \frac{1}{\tau}\right)^{-m} (2\tau-1)^{2k} \frac{m}{k} \binom{n}{m} \binom{k}{m-k} \mathfrak{Z}_F\left(k, \tau - \frac{1}{2}\right) \qquad (13)\end{aligned}$$

and

$$\mathfrak{Z}_F\left(k, \tau - \frac{1}{2}\right) = \sum_{m=1}^{k} \sum_{n=1}^{m} (-1)^{n+1} \left(2 - \frac{1}{\tau}\right)^{m} (2\tau-1)^{-2k} \binom{2k-m-1}{k-1} \binom{m}{n} \lambda_F(n,\tau). \qquad (14)$$

Proof. Let us recall that

$$\lambda_F(n,\tau) = n\tau\mathcal{Z}_F^*(1,\tau) + \sum_{m=2}^{n}(-1)^{m+1}\binom{n}{m}\tau^m\mathcal{Z}_F(m,\tau). \tag{15}$$

Since the zeros of $F \in S_{\mathbb{R}}^{\sharp\flat}$ come in pairs $\rho_k = 1/2 \pm i\tau_k$ with $\Re(\tau_k) > 0$, then Voros identity holds for the class $S_{\mathbb{R}}^{\sharp\flat}$; and with $t = \tau - \frac{1}{2}$, for $m \geq 2$ we get

$$\mathcal{Z}_F(m,\tau) = m\sum_{m/2\leq k\leq m}(-1)^{m-k}\binom{k}{m-k}\left(2\left(\tau-\frac{1}{2}\right)\right)^{2k-m}\frac{\mathfrak{Z}_F(k,\tau-\frac{1}{2})}{k}. \tag{16}$$

Then, (15) and (16) yield

$$\begin{aligned}
&\lambda_F(n,\tau)\\
=&n\tau\mathcal{Z}_F^*(1,\tau) + \sum_{m=2}^{n}(-1)^{m+1}m\binom{n}{m}\tau^m\sum_{m/2\leq k\leq m}(-1)^{m-k}\binom{k}{m-k}\left(2\left(\tau-\frac{1}{2}\right)\right)^{2k-m}\frac{\mathfrak{Z}_F(k,\tau-\frac{1}{2})}{k}\\
=&n\tau\mathcal{Z}_F^*(1,\tau) + \sum_{m=2}^{n}\sum_{m/2\leq k\leq m}(-1)^{2m-k+1}m\binom{n}{m}\binom{k}{m-k}\tau^m\left(2\left(\tau-\frac{1}{2}\right)\right)^{2k-m}\frac{\mathfrak{Z}_F(k,\tau-\frac{1}{2})}{k}.
\end{aligned}$$

Let us prove (14). By (7), for $m \geq 2$ one has

$$\mathcal{Z}_F(m,\tau) = \frac{1}{\tau^m}\sum_{n=1}^{m}(-1)^{n+1}\binom{m}{n}\lambda_F(n,\tau).$$

From Voros identity with $t = \tau - \frac{1}{2}$ (which holds also for the class $S_{\mathbb{R}}^{\sharp\flat}$), we obtain

$$\mathfrak{z}_F\left(k,\tau-\frac{1}{2}\right) = \binom{2k-2}{k-1}\left(2\left(\tau-\frac{1}{2}\right)\right)^{-2k+1}\mathcal{Z}_F^*(1,\tau) + \sum_{m=2}^{k}\binom{2k-m-1}{k-1}\left(2\left(\tau-\frac{1}{2}\right)\right)^{-2k+m}\mathcal{Z}_F(m,\tau).$$

Let us note that from (5) and the definition of the τ-Li coefficients, we have $\mathcal{Z}_F^*(1,\tau) = \frac{1}{\tau}\lambda_F(1,\tau)$. Then

$$\begin{aligned}\mathfrak{z}_F\left(k,\tau-\frac{1}{2}\right) =& \binom{2k-2}{k-1}\left(2\left(\tau-\frac{1}{2}\right)\right)^{-2k+1}\frac{1}{\tau}\lambda_F(1,\tau) \\ &+ \sum_{m=2}^{k}\binom{2k-m-1}{k-1}\left(2\left(\tau-\frac{1}{2}\right)\right)^{-2k+m}\frac{1}{\tau^m}\sum_{n=1}^{m}(-1)^{n+1}\binom{m}{n}\lambda_F(n,\tau) \\ =& \sum_{m=1}^{k}\binom{2k-m-1}{k-1}\left(2\left(\tau-\frac{1}{2}\right)\right)^{-2k+m}\frac{1}{\tau^m}\sum_{n=1}^{m}(-1)^{n+1}\binom{m}{n}\lambda_F(n,\tau) \\ =& \sum_{m=1}^{k}\sum_{n=1}^{m}(-1)^{n+1}\left(2-\frac{1}{\tau}\right)^{m}(2\tau-1)^{-2k}\binom{2k-m-1}{k-1}\binom{m}{n}\lambda_F(n,\tau).\end{aligned}$$

Special values at negative integers s

Let us note that

$$\mathfrak{z}_F(-m,0) = \frac{(-1)^m}{2}\mathcal{Z}_F(-2m,1/2).$$

Moreover, from the definition of $\mathfrak{z}_F(s,t)$ one has

$$\mathfrak{z}_F(-m,t) = \sum_{k=1}^{\infty}\tau_k^m\left(1+\frac{t^2}{\tau_k^2}\right) = \sum_{l=0}^{m}\binom{m}{l}\mathfrak{z}_F(-(m-l),0)t^{2l}.$$

Hence

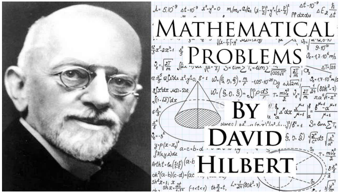
$$\mathfrak{z}_F(-m,t) = \frac{1}{2}\sum_{l=0}^{m}(-1)^{m-l}\binom{m}{l}\mathcal{Z}_F(-2(m-l),1/2)t^{2l}. \tag{17}$$

For a fixed integer n, the function $\mathcal{Z}_F(-n,z)$ has an analytic continuation to the whole z-plane (Odzak and Smajlovic, 2011). Hence, with $z=1/2$ and a fixed integer $n=2(m-l)$, for $(m-l)\geq 1$ we have

$$\begin{aligned}\mathcal{Z}_F(-2(m-l),1/2)) &= \frac{1}{2(2(m-l)+1)}H_F(2(m-l)+1)+\frac{m_F}{2^{2(m-l)-1}}\\ &+\frac{1}{2(2(m-l)+1)}\sum_{k=0}^{2(m-l)}\binom{2(m-l)+1}{k}H_F(k)\left(\frac{1}{2}\right)^{2(m-l)+1-k},\end{aligned} \tag{18}$$

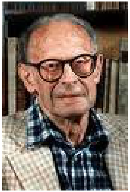
where for an integer $n\geq 0$ the numbers $H_F(n)$ are defined by $H_F(n)=2\sum_{j=1}^{r}\frac{B_n(\mu_j)}{\lambda_j^{n-1}}$ with $B_n(x)$ is the nth Bernoulli polynomial. The numbers $H_F(n)$ are called the H-invariants. By inserting (18) into (17), for a fixed integer m we obtain

$$\mathfrak{z}_F(-m,t) = \frac{(-1)^m m_F}{4^m}\left(1-4t^2\right)^m+\frac{(-1)^m}{4}\sum_{l=0}^{m}\binom{m}{l}\frac{H_F(2(m-l)+1)}{2(m-l)+1}(-t^2)^l$$



If I were to awaken after having slept for a thousand years, my first question would be: has the Riemann hypothesis been proven?

" Si je me réveillais après avoir dormi mille ans, ma première question serait: l'hypothèse de Riemann est-elle prouvée?"



André Weil

When I was young, I hoped to demonstrate the Riemann hypothesis. When I got a little older, I still had hope that I could read and understand a proof of the Riemann hypothesis. Now, I would be happy to learn that there is a proof.

"Quand j'étais jeune, j'espérais démontrer l'hypothèse de Riemann. Quand je suis devenu un peu plus vieux, j'ai encore eu l'espoir de pouvoir lire et comprendre une démonstration de l'hypothèse de Riemann. Maintenant, je me contenterais bien d'apprendre qu'il en existe une démonstration".

Thank you.

Merci pour votre attention.