Independence of notions from dynamics: a descriptive set theoretic approach

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W. Schmidt proved that $\mathcal{N}(p) = \mathcal{N}(q)$ iff $\frac{\log p}{\log q} \in \mathbb{Q}$.

Informally, there is an idea that the properties of being normal in base p and base q are "independent" if $\frac{\log p}{\log q} \notin \mathbb{Q}$.

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It is known that the Hausdorff dimension of $\mathcal{N}(p)\backslash\mathcal{N}(q)$ is equal to 1 under this condition. A motivation to prove a result like this involving two sets of full measure is to argue that they're as separated as possible under the circumstances and can thus be thought of in some ways as being "independent".

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I want to note the very strong result of AD Pollington. Let R and S be two subsets of $\mathbb{N}_{\geq 2}$ with the property that

$$r \in R, s \in S \implies \frac{\log r}{\log s} \notin \mathbb{Q}.$$

Then,

$$\bigcap_{r\in R} \mathcal{N}(r)\backslash \bigcup_{s\in S} \mathcal{N}(s)$$

has full Hausdorff dimension.

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Time for a wall of abstraction...

The Borel Hierarchy

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Examples:

$$\mathbb{R}, 2^{\mathbb{N}}, b^{\mathbb{N}}, 2^{\mathbb{N}\times\mathbb{N}}$$

The Borel Hierarchy

In any topological space X, the collection of Borel sets $\mathcal{B}(X)$ is the smallest σ-algebra containing the open sets. They are stratified into levels, the Borel hierarchy, by defining $\mathbf{\Sigma}^0_1$ = the open sets, $\mathbf{\Pi}^0_1 = \neg\mathbf{\Sigma}^0_1 = \{X - A \colon A \in \mathbf{\Sigma}^0_1\}$ = the closed sets, and for $\alpha < \omega_1$ we let $\mathbf{\Sigma}^0_\alpha$ be the collection of countable unions $A = \bigcup_n A_n$ where each $A_n \in \mathbf{\Pi}^0_{\alpha_n}$ for some $\alpha_n < \alpha$. We also let $\mathbf{\Pi}^0_\alpha = \neg\mathbf{\Sigma}^0_\alpha$. Alternatively, $A \in \mathbf{\Pi}^0_\alpha$ if $A = \bigcap_n A_n$ where $A_n \in \mathbf{\Sigma}^0_{\alpha_n}$ where each $\alpha_n < \alpha$. We also set $\mathbf{\Delta}^0_\alpha = \mathbf{\Pi}^0_\alpha \cap \mathbf{\Sigma}^0_\alpha$, in particular $\mathbf{\Delta}^0_1$ is the collection of clopen sets. For any topological space, $\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$. All of the collections $\mathbf{\Delta}^0_\alpha$, $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ are pointclasses, that is, they are closed under inverse images of continuous functions.

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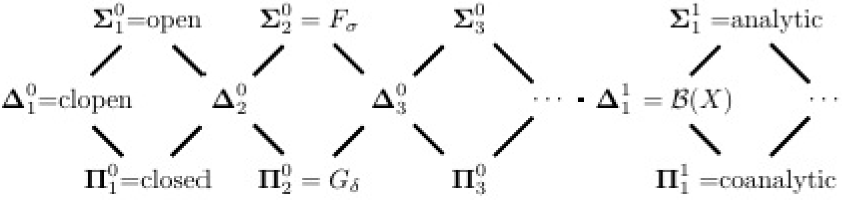
For example, $\mathbf{\Sigma}^0_2$ consists of F_σ sets and $\mathbf{\Pi}^0_2$ consists of G_δ sets. $\mathbf{\Pi}^0_3$ contains the sets which are intersections of F_σ sets.

The Borel Hierarchy

A fundamental result of Suslin says that in any Polish space $\mathcal{B}(X) = \mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$, where $\mathbf{\Pi}^1_1 = \neg\mathbf{\Sigma}^1_1$, and $\mathbf{\Sigma}^1_1$ is the pointclass of continuous images of Borel sets. Equivalently, $A \in \mathbf{\Sigma}^1_1$ iff A can be written as $x \in a \leftrightarrow \exists y \ (x, y) \in B$ where $B \subseteq X \times Y$ is Borel (for some Polish space Y). Similarly, $A \in \mathbf{\Pi}^1_1$ iff it is of the form $x \in A \leftrightarrow \forall y \ (x, y) \in B$ for a Borel B. The $\mathbf{\Sigma}^1_1$ sets are also called the *analytic* sets, and $\mathbf{\Pi}^1_1$ the *co-analytic sets*. We also have $\mathbf{\Sigma}^1_1 \neq \mathbf{\Pi}^1_1$ for any uncountable Polish space.

The Borel Hierarchy



The Borel Hierarchy

A basic fact is that for any uncountable Polish space X, there is no collapse in the levels of the Borel hierarchy, that is, all the various pointclasses $\boldsymbol{\Delta}^0_\alpha$, $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$, for $\alpha < \omega_1$, are all distinct. Thus, these levels of the Borel hierarch can be used to calibrate the descriptive complexity of a set. We say a set $A \subseteq X$ is $\boldsymbol{\Sigma}^0_\alpha$ (resp. $\boldsymbol{\Pi}^0_\alpha$) *hard* if $A \notin \boldsymbol{\Pi}^0_\alpha$ (resp. $A \notin \boldsymbol{\Sigma}^0_\alpha$). This says A is "no simpler" than a $\boldsymbol{\Sigma}^0_\alpha$ set. We say A is $\boldsymbol{\Sigma}^0_\alpha$-*complete* if $A \in \boldsymbol{\Sigma}^0_\alpha - \boldsymbol{\Pi}^0_\alpha$, that is, $A \in \boldsymbol{\Sigma}^0_\alpha$ and A is $\boldsymbol{\Sigma}^0_\alpha$ hard. This says A is exactly at the complexity level $\boldsymbol{\Sigma}^0_\alpha$. Likewise, A is $\boldsymbol{\Pi}^0_\alpha$-complete if $A \in \boldsymbol{\Pi}^0_\alpha - \boldsymbol{\Sigma}^0_\alpha$.

The Borel Difference Hierarchy

Moreover, A is in $D_2(\mathbf{\Pi}^0_\alpha)$ if $A = B \setminus C$, where $B, C \in \mathbf{\Pi}^0_\alpha$. A is $D_2(\mathbf{\Pi}^0_\alpha)$-*hard* if $X \setminus A \notin D_2(\mathbf{\Pi}^0_\alpha)$ and A is $D_2(\mathbf{\Pi}^0_\alpha)$-*complete* if it is $D_2(\mathbf{\Pi}^0_\alpha)$ and $D_2(\mathbf{\Pi}^0_\alpha)$-hard.

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Similarly, A is in $D_2(\mathbf{\Sigma}^0_\alpha)$ if $A = B \backslash C$, where $B, C \in \mathbf{\Sigma}^0_\alpha$. A is $D_2(\mathbf{\Sigma}^0_\alpha)$-*hard* if $X \backslash A \notin D_2(\mathbf{\Sigma}^0_\alpha)$ and A is $D_2(\mathbf{\Sigma}^0_\alpha)$-*complete* if it is $D_2(\mathbf{\Sigma}^0_\alpha)$ and $D_2(\mathbf{\Sigma}^0_\alpha)$-hard.

Examples

In $\mathbb{R}$, $(a, b) \in \boldsymbol{\Sigma}^0_1$, $[a, b] \in \boldsymbol{\Pi}^0_1$, and $\mathbb{R} \in \boldsymbol{\Delta}^0_1$.

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$\mathbb{Q} \in \mathbf{\Sigma}^0_2, \mathbb{R}\backslash\mathbb{Q} \in \mathbf{\Pi}^0_2$, and $\emptyset = \mathbb{Q} \cap \mathbb{R}\backslash\mathbb{Q} \in \mathbf{\Delta}^0_2$.

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Note that $\mathbb{Q}$ is $\boldsymbol{\Sigma}^0_2$-complete, $\mathbb{R}\backslash\mathbb{Q}$ is $\boldsymbol{\Pi}^0_2$-complete, but $\emptyset \in \boldsymbol{\Delta}^0_1$.

Let $X = C([0, 1])$ with the sup norm. If $S = \{f \in X : f$ is nowhere differentiable$\}$, then $S \in \boldsymbol{\Pi}^1_1\backslash\boldsymbol{\Sigma}^1_1$ (R. D. Mauldin 1979).

Examples with normal numbers

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Let

$$\mathcal{N}^{\perp}(b) = \{y \colon \forall x \in \mathcal{N}(b)\ (x+y) \in \mathcal{N}(b)\}.$$

be the set of numbers that preserve normality in base b under addition. The set $\mathcal{N}^{\perp}(b)$ is $\mathbf{\Pi}_3^0$-complete (Airey, Jackson, M. 2022).

From topological dynamics

A topological dynamical system is a pair (X, f) where X is a compact metric space and $f \in C(X, X)$ is a continuous map of X to itself. Limit sets and backward limit sets provide some of the most important tools in understanding the behavior of a topological dynamical system, since they provide information about the long-term behavior of the orbits of the system. One notion, in particular, of a backward limit set is the notion of a special α-limit set, which has played an important role in one-dimensional dynamics.

From topological dynamics

The α-limit set of a backward orbit, denoted $\alpha((x_n)_{n=0}^{\infty})$, consists of all accumulation points of a single *backward orbit*, i.e. a sequence $(x_n)_{n\in\mathbb{N}}$, where $f(x_{n+1}) = x_n$ for all n. The special α-limit set of a point, denoted $s\alpha(x)$, is the union $\bigcup \alpha((x_n)_{n=0}^{\infty})$ taken over all backward orbits of x, i.e. sequences $(x_n)_{n\in\mathbb{N}}$ such that $f(x_{n+1}) = x_n$ for all n and $x_0 = x$.

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Kolyada, Misiurewicz, and Snoha pointed out that special α-limit sets need not be closed, and asked whether they are necessarily Borel or even analytic. The difficulty arises when x has uncountably many backward orbit branches, since we are then taking an uncountable union of their (closed) accumulation sets. If $X = [0, 1]$, then $s\alpha(x)$ is always F_σ and G_δ.

From topological dynamics

This question was answered by Jackson, M., and Roth. Clearly, if $X = [0,1]$, then $s\alpha(x)$ must always be Borel as it is both F_σ and G_δ.

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We only need to consider $X = [0,1]^2$. It is possible to cook up a topological dynamical system (X, f) and a point $x \in X$ where $s\alpha(x)$ is co-analytic complete (hence, not Borel).

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Based on 26 pages of hand written notes (that surely have a few mistakes in them), it seems likely that for any countable ordinal ω, there exists a topological dynamical system on the square and points x and y where $s\alpha(x)$ is $\mathbf{\Sigma}^0_\omega$-complete and $s\alpha(y)$ is $\mathbf{\Pi}^0_\omega$-complete.

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Interpretation: you can say a lot about special α limit sets on the interval, but once you get to the square, there aren't any "nice hidden theorems".

Continued fractions

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Jackson, M., and Vandehey showed that the set of numbers that are continued fraction normal, but not normal in a fixed base b is $D_2(\mathbf{\Pi}_3^0)$-complete. The set of numbers that is continued fraction normal, but not normal in any base b is $D_2(\mathbf{\Pi}_3^0)$-hard.

Base b expansions

From the same paper, it follows that $\mathcal{N}(p) \backslash \mathcal{N}(q)$ is $D_2(\mathbf{\Pi}_3^0)$-complete if p and q are relatively prime.

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There can be no $\boldsymbol{\Sigma}_3^0$ set A such that $\mathcal{N}(p) \cap A = \mathcal{N}(q)$. This leads to some interesting corollaries.

Applications

Many naturally occurring sets of reals A are defined by conditions which result in them being $\mathbf{\Sigma}^0_3$ sets. Examples include countable sets, co-countable sets, the class BA of *badly approximable* numbers (which is a $\mathbf{\Sigma}^0_2$ set), the Liouville numbers (which is a $\mathbf{\Pi}^0_2$ set), and the set of $x \in [0, 1]$ where a particular continuous function $f : [0, 1] \to \mathbb{R}$ is not differentiable. Thus, for example, the theorem that being normal in base 2 and badly approximable is equivalent to being normal in base 3 is false. And many other examples.

Setup

If $\mathcal{A}$ is a finite or countable set, which we call the *alphabet*, then the *full shift space* over $\mathcal{A}$ is the pair $(\mathcal{A}^\omega, \sigma)$ where $\mathcal{A}^\omega$ is endowed with the product topology induced by the discrete topology on $\mathcal{A}$, and σ stands for the shift map, which is given for $(x_n)_{n\in\omega} \in \mathcal{A}^\omega$ by $\sigma(x)_n = x_{n+1}$. By a *subshift* of $\mathcal{A}^\omega$ (or *over* $\mathcal{A}$) we mean a pair (X, σ), where X is a nonempty closed shift-invariant subset of $\mathcal{A}^\omega$, and σ is the shift map restricted to X.
Recall that a Borel probability measure μ on $\mathcal{A}^\omega$ is *shift-invariant* if $\mu(A) = \mu(\sigma^{-1}(A))$ for every Borel set $A \subset \mathcal{A}^\omega$. We say that μ is a shift-invariant measure is an invariant measure for a subshift X if X contains the support of μ. An invariant measure is *ergodic* if for every Borel set $A \subset \mathcal{A}^\omega$ the condition $\sigma^{-1}(A) \subset A$ implies $\mu(A) \in \{0, 1\}$. For $n \geq 1$ and a block $w \in \mathcal{A}^n$, by $[w]$ we denote the cylinder consisting of those $x \in \mathcal{A}^\omega$ with $x_i = w_i$ for $1 \leq i \leq n$.

Setup

We say that a finite block $w \in \mathcal{A}^n$ *appears* in $x \in \mathcal{A}^\omega$ at the position $\ell \in \omega$ if $x_{\ell+i-1} = w_i$ for each $1 \le i \le n$. Let $e(w, x, N)$ be the number of times w appears in x at a position $\ell < N$. Let X be a subshift over $\mathcal{A}$ and μ be its invariant measure. A point $x \in X$ is *generic* for μ if for every finite block $w \in \mathcal{A}^n$ the set of positions at which w appears in x has the frequency equal to the measure of the set of all sequences starting with w, that is, if

$$\lim_{N \to \infty} \frac{e(w, x, N)}{N} = \mu([w]),$$

where $[w] = \{z \in \mathcal{A}^\omega : z_0 = w_1, \ldots, z_{n-1} = w_n\}$. By the shift-invariance of μ the measure of $[w]$ is equal to the μ-probability of the occurrence of w at any fixed position $\ell \in \omega$, that is,

$$\mu([w]) = \mu(\{z \in \mathcal{A}^\omega : z_\ell = w_1, \ldots, z_{\ell+n-1} = w_n\}).$$

Setup

For a shift space $X \subseteq \mathcal{A}^\omega$ and integer $n \geq 1$, we write $\mathcal{L}_n(X) \subseteq \mathcal{A}^n$ for the set of n-blocks appearing in X, that is $w \in \mathcal{L}_n(X)$ if and only if there exists some $x \in X$ and $\ell \in \omega$ such that $x_{\ell+i-1} = w_i$ for all $1 \leq i \leq n$. The *length* of a block w over $\mathcal{A}$ is the number of symbols in w and it is denoted by $|w|$. We agree that $\mathcal{A}^0$ consists of a single element, called the *empty word*, that is, $\mathcal{A}^0$ contains only the unique block over $\mathcal{A}$ of length 0. By $\mathcal{A}^{<\omega}$ we denote the set of all finite blocks over $\mathcal{A}$ (including the empty word). We let $\mathcal{L}(X) = \bigcup_{n \geq 1} \mathcal{L}_n(X)$ and call $\mathcal{L}(X)$ the language of X.

Specification

A shift space X over an at most countable alphabet $\mathcal{A}$ has the *specification property* if there is a nonnegative integer N such that if $w_i \in \mathcal{L}(X)$ for $i = 1, \ldots, n$ then there are $v_i \in \mathcal{A}^N$ for $i = 1, \ldots, n-1$ such that $u = w_1 v_1 w_2 v_2 \ldots v_{n-1} w_n \in \mathcal{L}(X)$.

Specification

Let d_H stand for the normalised Hamming distance, that is, given two blocks $u = u_1 \dots u_n$ and $w = w_1 \dots w_n$ of equal length we set $d_H(u, w) = |\{1 \leq j \leq n : u_j \neq w_j\}|/n$.

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We say that a subshift X has the *right feeble specification* property if there exists a set $\mathcal{G} \subseteq \mathcal{L}(X)$ satisfying:

1. a concatenation of words in $\mathcal{G}$ stays in $\mathcal{G}$, that is, if $u, v \in \mathcal{G}$, then $uv \in \mathcal{G}$;
2. for any $\epsilon > 0$ there is an $N = N(\epsilon)$ such that for every $u \in \mathcal{G}$ and $v \in \mathcal{L}(X)$ with $|v| \geq N$, there are $s, v' \in \mathcal{A}^{<\omega}$ satisfying $|v'| = |v|$, $0 \leq |s| \leq \epsilon|v|$, $d_H(v, v') < \varepsilon$, and $usv' \in \mathcal{G}$.

Irregular set

Given $w \in \mathcal{L}(X)$ we define $I_w(X)$ to be the set of all $x \in X$ such that the set of positions at which w appears in x does not have a frequency, that is

$$\liminf_{N \to \infty} \frac{e(w, x, N)}{N} < \limsup_{N \to \infty} \frac{e(w, x, N)}{N}.$$

Let $I(X)$ be the *irregular set* for X, that is, the union of sets $I_w(X)$ over all $w \in \mathcal{L}(X)$. The *quasi-regular set* for X is the complement of $I(X)$, that is, $Q(X) = X \setminus I(X)$. Both sets are obviously Borel and belong to the third level of the Borel hierarchy.

Main Theorem

Note that we are considering subshifts which are not necessarily compact. It forces us to *assume* that there are at least two shift-invariant measures on X. This condition is automatically fulfilled if X is compact.

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Theorem Assume that $\mathcal{A}$ is at most countable and X is a subshift over $\mathcal{A}$ with the right feeble specification property. If X has at least two invariant measures, then for every shift-invariant measure μ on X the set of generic points G_μ is $\mathbf{\Pi}_3^0$-complete. Furthermore, the quasi-regular set $Q(X)$ is $\mathbf{\Pi}_3^0$-complete and the irregular set $I(X)$ is $\mathbf{\Sigma}_3^0$-complete.

Corollaries

Corollary The set of normal numbers for the b-ary expansions, β-expansions, regular continued fraction expansion, and generalized GLS expansions are all $\mathbf{\Pi}_3^0$-complete.

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Corollary If μ is a Borel probability measure invariant for the tent map T, then the set of points that generate μ (also known as as the statistical basin for μ) is a $\mathbf{\Pi}_3^0$-complete set. The set of irregular points is $\mathbf{\Sigma}_3^0$-complete.

Wadge reduction

Let X and Y be Polish spaces and let $A \subseteq X$ and $B \subseteq Y$ along with a continuous function $f : Y \to X$ where $f^{-1}(A) = B$. Then if B is $\boldsymbol{\Sigma}^0_\alpha$-complete (resp. $\boldsymbol{\Pi}^0_\alpha$-complete), then A is $\boldsymbol{\Sigma}^0_\alpha$-hard ($\boldsymbol{\Pi}^0_\alpha$-hard).

Wadge reduction

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The function f reduces the question of membership in A to membership in B.